On the Asymptotic Formula for the Number of Plane Partitions of Positive Integers

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A plane partition ω of the positive integer n is an array of non-negative integers

$$\begin{array}{cccc} \omega_{1,1} & \omega_{1,2} & \omega_{1,3} & \cdots \\ \omega_{2,1} & \omega_{2,2} & \omega_{2,3} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{array} \tag{1}$$

for which

$$\sum_{i,j} \omega_{i,j} = n$$

and the rows and columns are arranged in decreasing order:

$$\omega_{i,j} \geq \omega_{i+1,j}, \omega_{i,j} \geq \omega_{i,j+1}$$

for all $i, j \geq 1$. The non-zero entries $\omega_{i,j} > 0$ are called parts of ω. Sometimes, for the sake of brevity, the zeroes in the array (1) are deleted. For instance, the abbreviation

$$\begin{array}{ccc} 3 & 2 & 1 \\ 1 & 1 & \end{array}$$

is assumed to present a plane partition of $n = 8$ having 2 rows and 5 parts.

It seems that MacMahon was the first who introduced the idea of a plane partition; see [5]. He deals with the general problem of such partitions enumerating them by the size of

each part, number of rows and number of columns. These problems have been subsequently reconsidered by other authors who have developed methods, entirely different from those of MacMahon. For important references and more details in this direction we refer the reader to the monograph of Andrews [1; Chap. 11] and to the survey paper of Stanley [8; Chap. V].

Let $q(n)$ denote the total number of plane partitions of the integer $n \geq 1$. It turns out that

$$Q(x) = 1 + \sum_{n=1}^{\infty} q(n)x^n = \prod_{j=1}^{\infty} (1 - x^j)^{-j} \tag{2}$$

(see [1; Corollary 11.3] or [8; Corollary 18.2]). The asymptotic of $q(n)$ has been obtained by Wright [10]. It is given by the following formula:

$$\begin{aligned} q(n) &= \frac{[\zeta(3)]^{7/36}}{2^{11/36}\pi^{1/2}} n^{-25/36} \exp\{3[\zeta(3)]^{1/3}(n/2)^{2/3} + 2c\} \\ &\times [\gamma_0 + \sum_{h=1}^{r} \gamma_h n^{-2h/3} + O(n^{-2(r+1)/3})], \end{aligned} \tag{3}$$

where

$$\zeta(s) = \sum_{j=1}^{\infty} j^{-s}$$

is the Riemann's zeta function,

$$c = \int_0^{\infty} \frac{y \log y}{e^{2\pi y} - 1} dy \tag{4}$$

and γ_h are certain constants with explicitly given values. (3) implies the asymptotic equivalence

$$q(n) \sim \frac{\gamma_0[\zeta(3)]^{7/36}}{2^{11/36}\pi^{1/2}} n^{-25/36} \exp\{3[\zeta(3)]^{1/3}(n/2)^{2/3} + 2c\}, n \to \infty. \tag{5}$$

Many authors refer to (3) and (5) in their work on this subject. Wright [10; p. 179] states in his theorem that $\gamma_0 = 1$, however, at the the end of its proof his argument shows that the true value of γ_0 is $\gamma_0 = 3^{-1/2}$. In fact, the substitution $v = 1 + y$ [10; p. 188] that he makes in the integrals denoted by a_{2m} generates the factor

$$(3 + 2y)^{-(m+1/2)}, m = 0, 1, ... \tag{6}$$

in their integrands. Then, Wright shows how a_{2m} determine the coefficients γ_h in (3). He proves that $a_{2m} = (-1)^m b_{sm}/2\pi$, where b_{sm} is the coefficient of y^{2m} in the expansion of

$$(1 + y)^{2s+2m+13/12}(3 + 2y)^{-m-1/2}$$

(here s is a non-negative integer; see also the definition of the constants b_{sm} on p. 179 [10; formula (2.15)]). It turns out that if $h = 0$, one has to set in (6) $m = 0$, and thus by [10; formula (2.22)], γ_0 equals the constant term in the power series expansion of

$$(1 + y)^{2s+13/12}(3 + 2y)^{-1/2}.$$

Hence $\gamma_0 = 3^{-1/2}$.

The asymptotic equivalence (5) can be also obtained after a direct application of a theorem due to Meinardus [6] (see also [1; Chap. 6]) who has obtained the asymptotic behavior of the coefficients in the power series expansion of infinite products of the following form:

$$\prod_{j=1}^{\infty}(1-x^j)^{-a_j},$$

where $\{a_j\}_{j\geq 1}$ is given sequence of non-negative numbers. He introduced a scheme of assumptions on $\{a_j\}_{j\geq 1}$, which are satisfied by the generating function (2) of the numbers $q(n)$. Further generalizations of Meinardus' result are given in [3].

The object of this work is to show that Meinardus' theorem implies asymptotic formula (5) with $\gamma_0 = 3^{-1/2}$. This fact was also briefly mentioned in [4], where the limiting distribution of the trace of a random plane partition was studied. A sketch of the proof of (5) based on the fact that $Q(x)$ is an admissible function in the sense of Hayman [2] was presented there.

We start with the statement of Meinardus theorem [6] (see also [1; Chap. 6]) . Let

$$\prod_{j=1}^{\infty}\left(1-x^j\right)^{-a_j} = 1+\sum_{n=1}^{\infty} r(n)x^n,$$

where $x = e^{-\tau}$, $\Re e\ \tau > 0$. Consider also the associated Dirichlet series

$$D(s) = \sum_{j=1}^{\infty}\frac{a_j}{j^s}, \qquad s = \sigma + it. \tag{7}$$

Meinardus assumes that the following four conditions hold.

(i) $D(s)$ converges in the half-plane $\sigma > \alpha > 0$.

(ii) $D(s)$ can be analytically continued in the region $\Re e\ s \geq -C_0$ $(0 < C_0 < 1)$, where $D(s)$ is analytic except for a simple pole at $s = \alpha$ with residue A.

(iii) There exists a constant $C_1 > 0$ such that $D(s) = O(|t|^{C_1})$ uniformly in $\sigma \geq -C_0$ as $|t| \to \infty$.

(iv) Let

$$g(v) = \sum_{j=1}^{\infty} a_j x^j, \qquad x = e^{-v}, \tag{8}$$

where $v = y + 2\pi i w$ and y and w are real numbers. For $|\arg v| > \pi/4$ and $|w| \leq 1/2$ and for sufficiently small y, $g(v)$ satisfies

$$\Re e\ g(v) - g(y) \leq -C_2 y^{-\varepsilon}, \tag{9}$$

where $\varepsilon > 0$ is an arbitrary number, and $C_2 > 0$ is suitably chosen and may depend on ε.

Theorem [6] *If conditions (i) - (iv) hold, then*

$$r(n) = Cn^K \exp\left\{n^{\frac{\alpha}{\alpha+1}}\left(1+\frac{1}{\alpha}\right)[A\Gamma(\alpha+1)\zeta(\alpha+1)]^{\frac{1}{\alpha+1}}\right\}(1+O(n^{-K_1})),$$

where $\Gamma(\alpha)$ *is Euler's gamma function,*

$$C = e^{D'(0)}[2\pi(1+\alpha)]^{-1/2}[A\Gamma(\alpha+1)\zeta(\alpha+1)]^{\frac{1-2D(0)}{2+2\alpha}}, \tag{10}$$

$$K = \frac{D(0) - 1 - \alpha/2}{\alpha + 1}, \tag{11}$$

$$K_1 = \frac{\alpha}{\alpha + 1} \min\left(\frac{C_0}{\alpha} - \frac{\delta}{4}, \frac{1}{2} - \delta\right) \tag{12}$$

and δ is an arbitrary positive number.

In the case of plane partitions we have $\{a_j\}_{j\geq 1} = \{j\}_{j\geq 1}$ and thus by (7), $D(s) = \zeta(s-1)$. Thus, (i) - (iii) follow from well known properties of the zeta function (see e.g. [9; Chap. 13]). Moreover, in (ii) C_0 is an arbitrary constant within range $(0, 1)$, since the zeta function has analytical continuation in whole complex plane. It is easily seen that $\alpha = 2$, $A = 1$. To verify condition (iv) notice first that for $\{a_j\}_{j\geq 1} = \{j\}_{j\geq 1}$ the function $g(v)$ defined by (8) becomes

$$g(v) = \sum_{j=1}^{\infty} j \exp\left(-jy + 2\pi i j w\right).$$

Hence

$$\Re e g(v) - g(y) = \sum_{j=1}^{\infty} j e^{-jy}[\cos(2\pi j w) - 1] = \sum_{k=1}^{\infty} (-1)^k \frac{(2\pi w)^{2k}}{(2k)!} \sum_{j=1}^{\infty} j^{2k+1} e^{-jy}.$$

We simplify the inner sum here replacing its terms by the following derivatives:

$$j^{2k+1} e^{-jy} = (-1)^{2k+1} \frac{d^{2k+1}}{dy^{2k+1}}(e^{-jy}).$$

In this way we get

$$\begin{aligned} \Re e g(v) - g(y) &= \sum_{k=1}^{\infty} (-1)^{-(k+1)} \frac{(2\pi w)^{2k}}{(2k)!} \frac{d^{2k+1}}{dy^{2k+1}} (\sum_{j=1}^{\infty} e^{-jy}) \\ &= \sum_{k=1}^{\infty} (-1)^{-(k+1)} \frac{(2\pi w)^{2k}}{(2k)!} \frac{d^{2k+1}}{dy^{2k+1}} \left(\frac{1}{e^y - 1}\right). \end{aligned} \tag{13}$$

Further, to calculate the $(2k+1)$th derivative of $1/(e^y - 1)$ we apply Faa di Bruno's formula [7; Section 2.8]. Using also an identity for the Stirling numbers of the second kind $\sigma_n^{(m)}$ [7; Section 4.5], we find that:

$$\begin{aligned} \frac{d^{2k+1}}{dy^{2k+1}} \left(\frac{1}{e^y - 1}\right) &= \sum_{m=0}^{2k+1} \frac{(-1)^m m!}{(e^y - 1)^{m+1}} \sum_b \frac{(2k+1)! e^{my}}{(1!)^{b_1} b_1! (2!)^{b_2} \ldots [(2k+1)!]^{b_{2k+1}}} \\ &= \sum_{m=0}^{2k+1} \frac{(-1)^m m! e^{my}}{(e^y - 1)^{m+1}} \sigma_{2k+1}^{(m)} = (e^y - 1)^{-(2k+2)} \sum_{m=0}^{2k+1} (-1)^m m! e^{my} (e^y - 1)^{2k+1-m} \sigma_{2k+1}^{(m)}, \end{aligned}$$

where $\sum_b$ means that the summation is over all non-negative integers b_j that satisfy $n = \sum_j j b_j, m = \sum_j b_j$. It is clear that all terms in the last sum are close to zero if y is sufficiently small except the last one. It equals $(-1)^{2k+1}(2k+1)!\sigma_{2k+1}^{(2k+1)} = (-1)^{2k+1}(2k+1)!$. Therefore, after simple algebraic manipulations, we can rewrite (13) in the following form:

$$\begin{aligned} \Re e g(v) - g(y) &= \sum_{k=1}^{\infty} (-1)^k (2\pi w)^{2k} (e^y - 1)^{-(2k+2)} [2k + 1 + \psi_k(y)] \\ &= -\frac{(2\pi w)^2}{(e^y - 1)^4}[3 + \psi_1(y)] + \left\{\frac{(2\pi w)^4}{(e^y - 1)^6}[5 + \psi_2(y)] - \frac{(2\pi w)^6}{(e^y - 1)^8}[7 + \psi_3(y)]\right\} + \ldots, \end{aligned} \tag{14}$$

where

$$\psi_k(y) = \frac{(-1)^{2k+1}}{(2k)!} \sum_{m=0}^{2k} (-1)^m m! e^{my} (e^y - 1)^{2k+1-m} \sigma_{2k+1}^{(m)} = O(y)$$

as $y \to 0^+$. For sufficiently small y we also have $e^y - 1 = y + O(y^2)$. Moreover, the requirement $|\arg \tau| \geq \pi/4$ implies that $(2\pi w)^{2k} \geq y^{2k}$. Using these arguments, we conclude that the first term in (14) is $-(3/y^2)[1 + O(y)]$ as $y \to 0^+$. For the other terms in the curly brackets of (14) we get the estimate

$$\frac{(2\pi w)^{4l}}{y^{4l+2}} \left\{ 4l + 1 - \left(\frac{2\pi w}{y}\right)^2 [4l + 3 + O(y)] \right\} \leq \frac{(2\pi w)^{4l}}{y^{4l+2}} [-2 + O(y)] \leq -2y^{-2} + O(y^{-1}), \quad (15)$$

where $l = 1, 2, ...$ Suppose now that $\epsilon \in (0, 2]$. Estimate (15) implies that one can find certain constant C_2 such that

$$y^\epsilon [\Re e g(v) - g(y)] \leq -C_2 < 0.$$

If $\epsilon > 2$, then for any $\eta > 0$ and sufficiently small y we have $0 < y^{\epsilon-2} < \eta$. Therefore for such y's one can define a sequence of enough large positive integers $\{M_y\}$ satisfying the inequality $\eta/2M_y < y^{\epsilon-2}$, or equivalently, $-\eta > -2M_y y^{\epsilon-2}$. Now, estimate (15) shows that $-\eta$ exceeds the sum of M_y terms of the expansion of $y^\epsilon[\Re e g(v) - g(y)]$ and this sum in turn exceeds the sum of all terms of the expansion. Hence, for $\epsilon > 2$, we also obtain inequality (9) of condition (iv) with $C_2 = \eta$.

To calculate the constants $D(0)$ and $D'(0)$ we use the following functional equations (see [9; Chap. 13]):

$$\zeta(1 - z) = 2 \cos \frac{1}{2}\pi z \, (2\pi)^{-z} \Gamma(z) \, \zeta(z), \quad (16)$$

$$\Gamma(z) \, \zeta(z) = \int_0^\infty \frac{w^{z-1}}{e^w - 1} dw = (2\pi)^z \int_0^\infty \frac{w^{z-1}}{e^{2\pi w} - 1} dw. \quad (17)$$

Substituting $z = 2$ in (16) we get:

$$D(0) = \zeta(-1) = -\frac{1}{12}. \quad (18)$$

Next, differentiating (16) and (17) and combining the expressions of the corresponding derivatives, we obtain

$$\zeta'(1 - z) = \sin \frac{1}{2}\pi z \, 2^{-z} \pi^{1-z} \, \Gamma(z) \, \zeta(z) - 2 \cos \frac{1}{2}\pi z \int_0^\infty \frac{w^{z-1} \ln w}{e^{2\pi w} - 1} \, dw,$$

Substituting again $z = 2$, we find that

$$D'(0) = \zeta'(-1) = 2 \int_0^\infty \frac{w \ln w}{e^{2\pi w} - 1} \, dw = 2c, \quad (19)$$

where c is the constant defined by (4).

From (10) - (12), (18) and (19) we obtain

$$C = 2^{-11/36}(3\pi)^{-\frac{1}{2}}\,[\zeta(3)]^{7/36}\,e^{2c}, \quad K = -\frac{25}{36}, \quad K_1 = \frac{1}{3} - \varepsilon_1,$$

where $\varepsilon_1 > 0$. Notice that in (12) we assume that δ is sufficiently small and C_0 is close to 1. Then, applying Meinardus theorem, we obtain

$$q(n) \sim \frac{[\zeta(3)]^{7/36}}{2^{11/36}3^{1/2}\pi^{1/2}}\,n^{-25/36}\exp\left\{3[\zeta(3)]^{1/3}(n/2)^{2/3} + 2c\right\},$$

which shows in turn that Wright's formula [10] with $\gamma_0 = 3^{-1/2}$ is valid.

We also present some numerical computations of $q(n)$ based on formula (5) and the following recurrence:

$$n\,q(n) = \sum_{k=1}^{n} q(n-k)\,\beta_2(k).$$

Here $\beta_2(k)$ is the sum of the squares of the positive divisors of k (see [1; Section 14.6]). We used Maple to get the results summarized in the table below. The exact values of $q(n)$ are listed in its second column. The third and fourth columns contain the corresponding values of $q(n)$ computed when $\gamma_0 = 1$ and $\gamma_0 = 3^{-1/2}$, respectively.

n	$q(n)$	$\gamma_0 = 1$	$\gamma_0 = 3^{-1/2}$
10	500	910.69	525.79
100	$59\,206 \times 10^{12}$	$103\,709 \times 10^{12}$	$59\,876 \times 10^{12}$
1 000	$35\,426 \times 10^{80}$	$61\,507 \times 10^{80}$	$35\,511 \times 10^{80}$
10 000	$45\,075 \times 10^{397}$	$78\,113 \times 10^{397}$	$45\,098 \times 10^{397}$

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